NAVIER-STOKES EQUATIONS

It was shown that Euler equations for inviscid fluids can be derived using either the Reynolds transport theorem or Newton’s law of motion. For viscous fluids, the momentum equations can similarly be derived with an additional consideration of the shear stresses due to fluid viscosity.

In 1822, Claude-Louis Navier, a French scientist, derived momentum equations of fluids using the molecular theory of attraction and repulsion between neighboring molecules However, Navier could not explain the diffusion terms related to the fluid viscosity clearly. Later, in 1845, an Irish mathematician George Gabriel Stokes derived the same equations and explained that the diffusion terms are included due to the fluid viscosity. This is why we named two scholars for the momentum equations for viscous fluids.

The Navier-Stokes equations in the vector form are given by

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} = -\nabla p + \nu \nabla^2 \mathbf{V} \quad (1)$$

$$\nabla \cdot \mathbf{V} = 0 \quad (2)$$

which is for incompressible fluids. The Navier-Stokes equations probably contain all of turbulence. Yet it would be foolish to try to guess what it consequences are without looking at experimental facts. The phenomena are almost varied as in the realm of life (Frisch, 1995).

1. Stress Tensor

A commonly-used tensor notion is given by

τ_{ij}	**F**irst **F**ace ; **S**econd **S**tress direction

In other words, the first subscript denotes the direction of normal, and the second does the direction of action.

As seen in the figure below, for cubic element, there are 9 stresses, which are expressed by stress tensor τ_{ij} such as

$$\tau_{ij} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yz} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \tag{3}$$

(Q) Prove that the stress tensor is symmetric about diagonal. That is,

$\tau_{xy} = \tau_{yx}$, $\tau_{yz} = \tau_{zy}$, and $\tau_{zx} = \tau_{xz}$

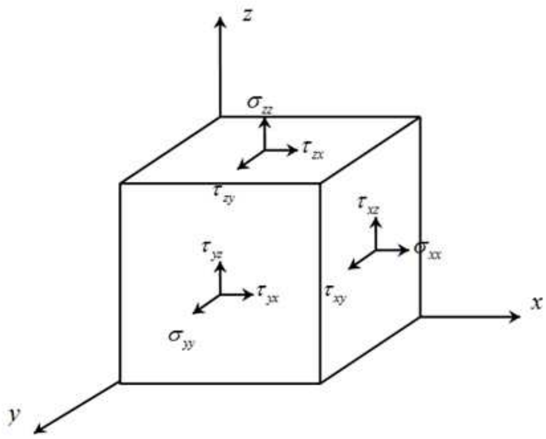


Figure 1. Stresses acting on a fluid element

2. The Navier-Stokes Equations

2.1 Force Balance

Consider the momentum balance of the cubic fluid element given below. If the fluid viscosity is considered, then the shear stress should be included in addition to the normal stresses or pressure we considered in the derivation of the Euler equations. Then, the external forces in the x – direction are given by

$$\delta \mathbf{F_x} = \left[\left(\sigma_x + \frac{\partial \sigma_x}{\partial x}\delta x\right) - \sigma_x\right]\delta y \delta z + \left[\left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\delta y\right) - \tau_{yx}\right]\delta x \delta z$$

$$+\left[\left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial y}\delta z\right) - \tau_{zx}\right]\delta y \delta x + f_x \rho \delta x \delta y \delta z \qquad (4)$$

where f_i is the body force component (per unit mass) in the i-th direction. The acceleration term is expressed as

$$\delta m\mathbf{a}_{\mathbf{x}} = \rho dxdydz \frac{Du}{Dt} \tag{5}$$

Therefore, in the x-direction, we have

$$\rho \frac{Du}{Dt} = \rho f_x + \left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right) \tag{6a}$$

Similarly, in the y- and z-directions, the momentum equations are given by

$$\rho \frac{Dv}{Dt} = \rho f_y + \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} \right) \tag{6b}$$

$$\rho \frac{Dw}{Dt} = \rho f_z + \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} \right) \tag{6c}$$

Note that, if the fluid is frictionless, all shear stresses vanish and only normal stresses remain in the equations. That is,

$$\tau_{xy} = \tau_{yz} = \tau_{zx} = 0$$

$$\sigma_x = \sigma_y = \sigma_z = -p$$

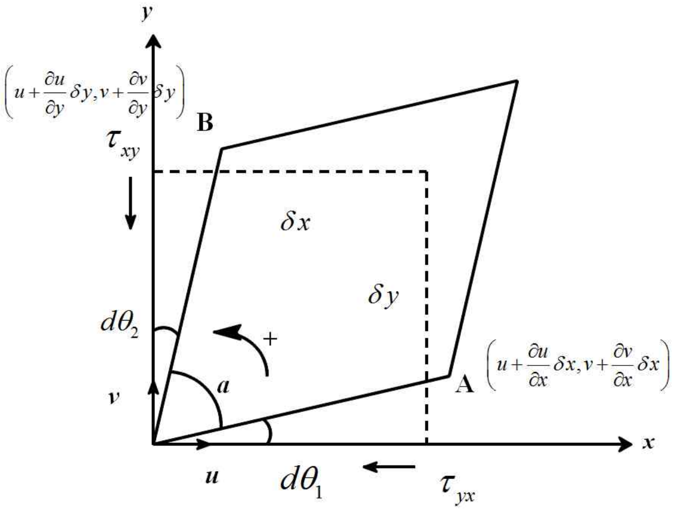


Figure 2. Deformation of a fluid element

2.2 Analysis of the Motion of a Fluid Element

(1) Shear Stress

The velocity components at points **A** and **B** are, respectively,

$$u_A = u + \frac{\partial u}{\partial x}\delta x \quad \text{and} \quad v_A = v + \frac{\partial v}{\partial x}\delta x$$

$$u_B = u + \frac{\partial u}{\partial y}\delta y \quad \text{and} \quad v_B = v + \frac{\partial v}{\partial y}\delta y$$

At the point **A**, the distance traveled in the *y*-direction during *dt* is

$$\Delta = \left[\left(v + \frac{\partial v}{\partial x}\delta x\right) - v\right]\delta t$$

which results in the angle of $d\theta_1 = \Delta / \delta x$. Then, the angular velocity of *dx* element is (positive angular velocity is defined in the counterclockwise direction)

$$\dot{\theta}_1 = \frac{d\theta_1}{dt} = \frac{1}{\delta t} \times \frac{\left(\frac{\partial v}{\partial x}\delta x\right)\delta t}{\delta x} = \frac{\partial v}{\partial x}$$

Similarly, the angular velocity of δy element is $\dot{\theta}_2 = -\partial u / \partial y$. The angular deformation or strain rate (ε_{ij}) is defined as the average of the difference in angular velocities of two originally perpendicular elements, i.e., $\varepsilon_{xy} = d\alpha / dt$. Here, $d\alpha = 1/2\left(d\theta_1 + \left(-d\theta_2\right)\right)$. So the rate at which *a* is changing (rate of angular deformation) is

$$\varepsilon_{xy} = \frac{1}{2}\left(\dot{\theta}_1 - \dot{\theta}_2\right) = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

Therefore, we have such shear stress component in the *x*-*y* plane as

$$\tau_{xy} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \tag{7a}$$

Generally, the shear stress components can be related to the velocity gradients by

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \tag{7b}$$

$$\tau_{zx} = \tau_{xz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \tag{7c}$$

(2) Normal Stress

In a viscous incompressible fluid, the normal stresses are the sum of the pressure force and a viscous force proportional to the coefficients of linear deformation:

$$\sigma_{xx} = -p + a\dot{\varepsilon}_x \tag{8}$$

where "*a*" represents a physical constant of a fluid medium and $\dot{\varepsilon}_x$ is the rate of elongation in the *x*-direction suffered by the fluid element. Stokes discovered that $a = 2\mu$. It is easy to deduce that $\dot{\varepsilon}_x = \partial u / \partial x$ from the fact that

$$du = \left(\frac{\partial u}{\partial x}\right) dx \tag{9}$$

which leads to $\dot{\varepsilon} = \nabla \mathbf{V}$. Thus, Eq.(8) can be cast in the following form:

$$\sigma_{xx} = -p + 2\mu \frac{\partial u}{\partial x} \tag{10}$$

However, in case of a viscous compressible fluid, the shear stresses remain the same, but the normal forces have to take into account the change of volume of the fluid particle. That is,

$$\sigma_x = -p + \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + 2\mu \frac{\partial u}{\partial x} \tag{11}$$

in which λ is a second coefficient of viscosity for a gas. From the kinetic theory of gases, it can be shown for a monoatomic gas that

$$3\lambda + 2\mu = 0 \tag{12}$$

which is called Stokes' relation. In practice this relationship is accurate enough for any kind of gas. Thus, Eq.(11) can be rewritten and equations for *y*- and *z*-directions can be deduced as

$$\sigma_x = -p - \frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + 2\mu\frac{\partial u}{\partial x} \tag{13a}$$

$$\sigma_y = -p - \frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + 2\mu\frac{\partial v}{\partial y} \tag{13b}$$

$$\sigma_z = -p - \frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + 2\mu\frac{\partial w}{\partial z} \tag{13c}$$

If one adds the above three equations, he has

$$\sigma_x + \sigma_y + \sigma_z = -3p - 2\mu\nabla\cdot\vec{V} + 2\mu\nabla\cdot\vec{V} = -3p$$

Therefore, we have

$$p = -\frac{1}{3}\left(\sigma_x + \sigma_y + \sigma_z\right)$$

The shear and normal stress components can be presented in the compact tensor notation by

$$\sigma_{ij} = -p\delta_{ij} - \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right) \quad (i, j, k = 1, 2, 3) \tag{14}$$

(3) Momentum Equations

Thus we have the Navier-Stokes equations for compressible fluids such as

$$\frac{Du}{Dt}=f_x-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{v}{3}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+v\nabla^2 u \tag{15a}$$

$$\frac{Dv}{Dt}=f_y-\frac{1}{\rho}\frac{\partial p}{\partial y}+\frac{v}{3}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+v\nabla^2 v \tag{15b}$$

$$\frac{Dw}{Dt}=f_z-\frac{1}{\rho}\frac{\partial p}{\partial z}+\frac{v}{3}\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+v\nabla^2 w \tag{15c}$$

For incompressible fluids, the Navier-Stokes equations are

$$\frac{Du}{Dt}=f_x-\frac{1}{\rho}\frac{\partial p}{\partial x}+v\nabla^2 u \tag{16a}$$

$$\frac{Dv}{Dt}=f_y-\frac{1}{\rho}\frac{\partial p}{\partial y}+v\nabla^2 v \tag{16b}$$

$$\frac{Dw}{Dt}=f_z-\frac{1}{\rho}\frac{\partial p}{\partial z}+v\nabla^2 w \tag{16c}$$

The Navier-Stokes equations enable engineers to analyze both laminar and turbulent flows. However, a fully deterministic approach is no longer possible in the case of turbulent motion because of the random nature of turbulence. Furthermore, in engineering practice, it is not always necessary to know the exact fine structure of the flow. This is why so many mathematical models averaged over time and space are preferred by engineers. Only the average values and the overall and statistical effects of turbulent fluctuations are sometimes of interest.

(Q) Average the Navier-Stokes equations over turbulence and obtain the Reynolds equations. Explain the role of the Reynolds stress terms clearly.

2.3 Concluding Remarks

For more than 50 years it has been recognized that our understanding of turbulent flows is very incomplete. A quotation attributed to Sir Horace Lamb in 1932 might still be appropriate:

“I am an old man now, and when I die and go to Heaven there are two matters on which I hope for enlightenment. One is electrodynamics and the other is the turbulent motion of fluids. And about the former I am rather optimistic.”

The fundamental equations of fluid dynamics are based on the following universal laws:

(1) conservation of mass

(2) conservation of momentum

(3) conservation of energy (1st law of thermodynamics)

In addition to these universal laws, it is necessary to establish relationships between fluid properties in order to close the system of equations. An example of such a relationship is the equation of state which relates the thermodynamic variables pressure, density, and temperature.

For incompressible fluids, the Navier-Stokes equations together with the continuity equation constitute four equations. Theoretically these equations may be solved because the number of unknowns are four (u_j and p). However, it should be remembered that the flow variables in the set of equations are instantaneous ones including chaotic effects, which are not of interest to many engineers. Furthermore, we have the following computational difficulty:

The exact equations describing the turbulent motion are known (the Navier-Stokes equations), and numerical procedures are available to solve these equations, but the storage capacity and speed of present computers is still not sufficient to allow a solution for any practically relevant turbulent flow. The reason is that the turbulent motion contains elements which are much smaller than the extent of the flow domain, typically of the order of 10^{-3} times smaller. To resolve the motion of these elements in a numerical procedure, the mesh size of the numerical grid would have to be even smaller; therefore at least 10^9 grid points would be necessary to cover the flow domain in three dimensions.

For compressible fluids (without external heat addition or body forces), three more dependent variables should be added to four for incompressible fluids. That is, they are the fluid density (ρ), internal energy per unit mass (e), and temperature (T). Therefore, two more equations are needed to close the system together with the continuity equation, three momentum equations, and energy equation. Generally, the two state equations such as $p = p(e, \rho)$ and $T = T(e, \rho)$

are employed. Examples of state equations are

$$p = \rho RT$$

$$e = c_v T$$

where R is the gas constant and c_v is the specific heat of constant volume which is a function of R. The former is the equation of state for perfect gas, whose inter-molecular forces are negligible.

References

Frisch, U. (1995). *Turbulence*. Cambridge University Press, Cambridge, Great Britain.

Hallback, D.S., Henningson, D.S., Johansson, A.V., and Alfredsson, P.H. (1996). *Turbulence and Transition Modelling*. Kluwer Academic Publishers, Dordrecht, The Netherland. (p. 270)

Schlichting, H. (1979). *Boundary Layer Theory* (Seventh Ed.). McGraw-Hill Book Company, New York, NY.

Problems

1. Parallel flow through a straight channel

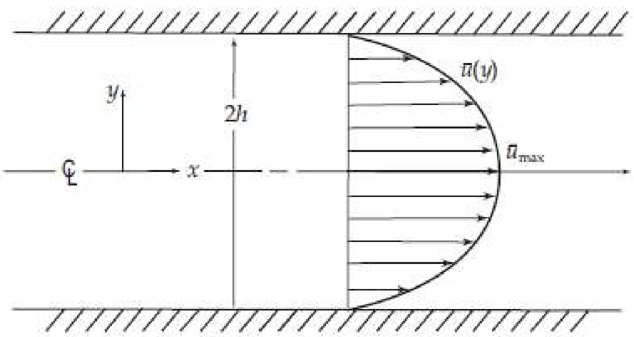
Consider 2D flow depicted in the figure below. For steady, incompressible flow, the continuity and momentum equations are given by, respectively,

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

Solve these equations analytically with appropriate boundary conditions.



2. Couette flow

Consider the flow between two parallel infinite plates. The upper plate is moving at a velocity U. The flow is two-dimensional, steady, incompressible.

(1) Obtain the governing equation(s).

(2) Give the proper boundary conditions for the governing equation(s).

(3) Solve these analytically.